

# A NEW YANG NUMBER AND CONSEQUENCES

DRAGOMIR Ž. ĐOKOVIĆ

**ABSTRACT.** Base sequences  $BS(m, n)$  are quadruples  $(A; B; C; D)$  of  $\{\pm 1\}$ -sequences,  $A$  and  $B$  of length  $m$  and  $C$  and  $D$  of length  $n$ , the sum of whose non-periodic auto-correlation functions is zero. Base sequences and some special subclasses of  $BS(n+1, n)$  known as normal and near-normal sequences,  $NS(n)$  and  $NN(n)$ , as well as T-sequences and orthogonal designs play a prominent role in modern constructions of Hadamard matrices. In our previous papers [3, 4] we have classified the near-normal sequences  $NN(s)$  for all even integers  $s \leq 32$  (they do not exist for odd  $s > 1$ ). We now extend the classification to the case  $s = 34$ . Moreover we construct the first example of near-normal sequences  $NN(36)$ . Consequently, we construct for the first time T-sequences of length 73. For all smaller lengths, T-sequences were already known. Another consequence is that 73 is a Yang number, and a few important consequences of this fact are given.

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## 1. PRELIMINARIES

A sequence  $A = a_1, a_2, \dots, a_m$ , of length  $m$ , is *binary* respectively *ternary* if  $a_i \in \{\pm 1\}$  respectively  $a_i \in \{0, \pm 1\}$ . We identify  $A$  with the polynomial  $A(z) = a_1 + a_2z + \dots + a_mz^{m-1}$ . The *norm* of a Laurent polynomial  $f(z)$  is defined as  $N(f) = f(z)f(z^{-1})$ . A quadruple  $(A; B; C; D)$  of binary sequences, with  $A$  and  $B$  of length  $m$  and  $C$  and  $D$  of length  $n$ , are *base sequences* if

$$N(A) + N(B) + N(C) + N(D) = 2(m + n).$$

The set of such base sequences is denoted by  $BS(m, n)$ . The *Base Sequence Conjecture* (BSC) asserts that  $BS(n+1, n) \neq \emptyset$  for all non-negative integers  $n$  (see [2]). It has been confirmed for  $n \leq 35$ . It is also well known that it holds when  $n$  is a *Golay number*, i.e., a number of the form  $2^a 10^b 26^c$  where  $a, b, c$  are nonnegative integers.

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The base sequences  $(A; B; C; D) \in BS(n+1, n)$  are *normal* respectively *near-normal* if  $a_i = b_i$  respectively  $a_i = (-1)^{i-1}b_i$  for all indexes  $i$  in the interval  $1 \leq i \leq n$ . Let  $NS(n)$  resp.  $NN(n)$  be the subset of  $BS(n+1, n)$  consisting of all normal resp. near-normal sequences. It is known that  $NS(n) \neq \emptyset$  when  $n$  is a Golay number. For positive integers  $n \leq 33$ ,  $NS(n) = \emptyset$  exactly for

$$n \in \{6, 14, 17, 21, 22, 23, 24, 27, 28, 30, 31, 33\}$$

(see [4]). It is known that  $NN(n) = \emptyset$  when  $n$  is odd. *Yang Conjecture* (YC) asserts that  $NN(n) \neq \emptyset$  when  $n$  is even. This has been confirmed for  $n \leq 34$  (see [4]).

A quadruple of ternary sequences  $(A; B; C; D)$ , all four of length  $n$ , are *T-sequences* if

$$N(A) + N(B) + N(C) + N(D) = n$$

and for each index  $i$ ,  $1 \leq i \leq n$ , exactly one of  $a_i, b_i, c_i, d_i$  is nonzero.

Let  $x_i$ ,  $i = 1, 2, \dots, u$ , be independent commuting variables. An  $n \times n$  matrix  $S$  all of whose entries belong to  $\{0, \pm x_1, \dots, \pm x_u\}$  and such that

$$SS^T = (s_1x_1^2 + \dots + s_ux_u^2)I_n$$

is an *orthogonal design*. Here the superscript  $T$  denotes transposition of matrices and  $I_n$  is the identity matrix. The symbols  $s_1, \dots, s_u$  are positive integers. The set of all such orthogonal designs is denoted by  $OD(n; s_1, \dots, s_u)$ .

Yang [6, Theorems 1 and 3] constructed two maps, known as *Yang multiplications*,

$$\begin{aligned} NS(s) \times BS(m, n) &\rightarrow TS((2s+1)(m+n)), \\ NN(s) \times BS(m, n) &\rightarrow TS((2s+1)(m+n)). \end{aligned}$$

(As pointed out in [4], there are two misprints in his Theorem 1.) A *Yang number* is an odd integer  $2s+1$  such that  $NS(s)$  or  $NN(s)$  is not empty.

## 2. MAIN RESULT AND CONSEQUENCES

We have carried out an exhaustive search for  $NS(n)$  when  $n = 34, 35$  and found that both are empty.

In our paper [3] we have introduced an equivalence relation, NN-equivalence, and a canonical form for near-normal sequences and we have enumerated the equivalence classes in  $NN(n)$  for even integers  $n \leq 30$ . For the classification in the case  $n = 32$  see [4]. We have now completed the classification in the case  $n = 34$ . The representatives  $(A; B; C; D)$  of the NN-equivalence classes of  $NN(34)$  are given in Table

1. The sums  $a, b, c, d$  of the sequences  $A, B, C, D$  are also recorded. The sequences are given in encoded form for the sake of compactness. The encoding scheme is explained in our papers [2, 3]. This table also contains the unique known example (up to NN-equivalence) of  $NN(36)$ , which we have constructed recently.

**Theorem 2.1.**  $NN(36) \neq \emptyset$ .

*Proof.* Direct verification. □

**Table 1: Near-normal sequences  $NN(n)$**

	$A \ \& \ B$	$C \ \& \ D$	$a, b, c, d$
$n = 34$			
1	076417646512321462	16738541372344337	7, 7, -2, 6
2	076535878535141762	17677852174231455	-5, 7, 0, 8
3	076782178767646231	17621532262576812	-5, 3, 10, -2
4	058214353712141461	11868756376664254	11, 3, -2, 2
5	053765656464871261	17765746348615187	1, 1, -6, 10
$n = 36$			
1	0764841234846532153	165154775335162126	3, -3, 8, 8

For the convenience of the reader we decode the above example of  $NN(36)$  and write it as a quadruple  $(A; B; C; D)$  of binary sequences. In fact we write + for +1 and - for -1.

$$\begin{aligned}
A &= + - + + - + + + - + - + + - - + + - - + \\
&\quad + + + + - - - - + + + - - - - - +; \\
B &= + + + - - - + - - - - - + + - - + + - - \\
&\quad + - + - - + - + + - + + - + - + -; \\
C &= + + - + - + - - - - - - + + + + + + - + \\
&\quad + + - + + + + + - - - + + + - +; \\
D &= + + + + + - + + + - - + + + - + - + - - \\
&\quad + - - + - + + - + + + - + - - +.
\end{aligned}$$

Let us list some consequences of the above results.

(i) Since  $NN(n) \subseteq BS(n+1, n)$ , we can update the status of the BSC and YC:  $BS(n+1, n) \neq \emptyset$  for  $n \leq 36$ , and  $NN(n) \neq \emptyset$  for even  $n \leq 36$ .

(ii) An odd (positive) integer  $n \leq 73$  is a Yang number if and only if  $n \neq 35, 43, 47, 55, 63, 67, 71$ . In order to rule out the integer 71, we need to use the fact that  $NS(35) = \emptyset$  mentioned above.

(iii) Our main result implies that there exist T-sequences  $TS(73)$  (see [5, Lemma 5.21]). Consequently, by [5, Theorem 3.6], there exists an orthogonal design  $OD(4t; t, t, t, t)$  for  $t = 73$ . Neither T-sequences nor T-matrices for  $n = 73$  were known previously, see the Remarks V 2.51 and V 8.47 in [1]. In spite of the claim made in Remark V 2.119.5 that an OD with the above parameters is known, we believe that this is not the case as we could not find such a result anywhere in the literature and our request for a reference failed.

(iv) One can plug into our OD any Williamson-type matrices of some order  $n$  to obtain Hadamard matrices of order  $4 \cdot 73 \cdot n$ . Since Williamson-type matrices are known for infinitely many odd orders  $n$ , we obtain infinitely many Hadamard matrices of order an odd multiple of 4. In particular, there exist Williamson-type matrices of order  $n = 61, 81, 83, 101$  (see [5, Table A.1]). Hence, there exist Hadamard matrices of order  $4 \cdot 73 \cdot n$  for the same values of  $n$ . These orders fall in the range covered by Table V 1.53 of [1], where it is indicated that no Hadamard matrices of these orders are known. However, we have recently discovered over 100 errors in this table. It turns out that Hadamard matrices of these four orders are in fact known. This will be discussed in more details elsewhere.

(v) For  $s = 2, 4, \dots, 34$ , the number of NN-equivalence classes in  $NN(s)$  is 1, 2, 2, 3, 8, 14, 11, 24, 20, 18, 32, 12, 3, 20, 9, 8, 5.

Our computer program performs an exhaustive search for near-normal sequences  $NN(s)$  for fixed  $s$ . The search is divided into 12 cases which can be run separately. For  $s = 36$ , we started to run them at different times according to the availability of machines. Six of the cases have completed within a month or two without finding any near-normal sequences. Only one of the remaining cases produced (after about 40 days) the example given in the above table.

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DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO, WATERLOO, ONTARIO, N2L 3G1, CANADA

*E-mail address:* djokovic@uwaterloo.ca